The thermal and two-particle stress-energy must be ill-defined on the 2-d Misner space chronology horizon

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We show that an analogue of the (four dimensional) image sum method can be used to reproduce the results, due to Krasnikov, that for the model of a real massless scalar field on the initial globally hyperbolic region IGH of two-dimensional Misner space there exist two-particle and thermal Hadamard states (built on the conformal vacuum) such that the (expectation value of the renormalised) stress-energy tensor in these states vanishes on IGH. However, we shall prove that the conclusions of a general theorem by Kay, Radzikowski and Wald still apply for these states. That is, in any of these states, for any point b on the Cauchy horizon and any neighbourhood N of b, there exists at least one pair of non-null related points $(x, x') \in (N \cap IGH) \times (N \cap IGH)$ such that (a suitably differentiated form of) its two-point function is singular. (We prove this by showing that the two-point functions of these states share the same singularities as the conformal vacuum on which they are built.) In other words, the stress-energy tensor in any of these states is necessarily ill-defined *on* the Cauchy horizon.

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I. INTRODUCTION

The question whether or not an arbitrarily advanced civilization could possibly manufacture a time machine is still open. It has been argued that such a construction may be impossible, since the Cauchy/chronology horizon arising in any such attempt would be semi-classically unstable, see [1] and references therein. It was thought that the nature of the semi-classical instability would be manifested by a divergence in (the expectation value of the renormalised) stress-energy tensor as the Cauchy horizon is approached. In fact, a simple example of such a divergence in the case of a real scalar field on Misner space [3] was used by Hawking [2] to argue that the divergence may be universal, that is that a similar statement may hold in any spacetime with a (compactly generated) Cauchy horizon for any physically acceptable initial state. We know now that this is not the case: It has been shown by Sushkov [5] in the context of a massless automorphic field on four dimensional Misner space [6] that there exist Hadamard states for which the stress-energy tensor vanishes on the initial globally hyperbolic region. Also, Krasnikov [7] has given examples of two-particle and thermal Hadamard [8] states (built on the conformal vacuum) for which the stress-energy tensor of a real massless scalar field on two-dimensional Misner space vanishes (or is bounded) on the initial globally hyperbolic region. (See also the earlier work of Sushkov [4].) Additionally, Boulware [10] in the context of Gott space and Hiscock and Tanaka [9] in the context of Grant space [11] have shown that for a sufficiently massive real scalar field the stress-energy tensor is bounded on the initial globally hyperbolic region. Also, Visser [12] has argued that with a suitable wormhole configuration – the Roman ring – the stress-energy tensor can be made arbitrarily small all the way up to the Cauchy horizon.

On the other hand, it has been shown rigorously by Kay, Radzikowski and Wald (KRW) [14] by using general theorems on 'propagation of singularities' [13] that for the model consisting of a real linear scalar field on a four dimensional spacetime with compactly generated Cauchy horizon (or on a spacetime which arises as the product of a Riemannian manifold of dimension $4-d$ with a spacetime with compactly generated Cauchy horizon with dimension $d < 4$) the stress-energy tensor for any Hadamard state is necessarily ill-defined on the horizon. (Technically, this ill-definedness is expressed in terms of a singularity property of the state's two-point function. In the present paper, it will be expressed in terms of a singularity property of the state's point-split stress-energy tensor.) By an obvious *reductio ad absurdum* argument this result can be given the interpretation that any spacetime with a compactly generated Cauchy horizon together with a real linear scalar field cannot arise as a solution of semi-classical gravity. (Suppose such a solution existed, then the stress-energy tensor would be well-defined at every point of the spacetime!) Note that

if it were permissible to assume continuity of the stress-energy tensor, one could argue that when the stress-energy tensor vanishes on the initial globally hyperbolic region then it must be defined and vanish on the Cauchy horizon. The KRW-theorem shows, instead, that the stress-energy tensor is not defined there and of course one conclusion from this is that the assumption of continuity cannot be permissible.

We should remark here that, even though the KRW-theorem by the above *reductio ad absurdum* argument can be given the status of a no-go theorem for semi-classically describable time machines, there still remains the possibility that a full theory of quantum gravity might still allow time machines to exist in a (yet unknown) sense which is not describable semi-classically. In fact, Visser [15] has argued for an invariant notion of *reliability horizon* as the future boundary of the region consisting of closed spacelike geodesics with length equal to or shorter than the Planck length and argues that we should not trust semi-classical gravity in the region to the future of the reliability horizon. Visser emphasizes that the very existence of models where the stress-energy can still be made small up to the Cauchy horizon of course prevents us from ruling out the possibility of such non-semi-classically describable time machines. However, it is not at all clear what a time-machine would be once one loses the notion of a classical spacetime, and even if there was such a notion, what it would mean for a physical "observer" to pass through such a time-machine. Thus, in the absence of a clearer understanding of quantum gravity, a strong case can still be made for identifying "time-machine" with "semi-classically describable time-machine" whereupon the KRW-theorem may be interpreted as a no-go theorem.

We remark that the KRW-theorem does not explicitly apply to Sushkov's [5] model consisting of a massless automorphic field on four dimensional Misner space, but it has been proved [16] in an elementary way that the conclusions of the KRW-theorem apply to Sushkov's model. Furthermore, the examples in two-dimensions by Krasnikov [7] are of a slightly different nature from the four dimensional examples, since the notion of Hadamard states on a two-dimensional spacetime is different from the corresponding notion in four dimensions and because of infrared divergences, in the massless theory, only derivatives of the field make sense [8]. Therefore, the KRW-theorem cannot be used immediately to make a statement on the nature of the stress-energy tensor on the Cauchy horizon in a two-dimensional spacetime. However, we shall in this paper prove, with a similar method to that used in [16], that the stress-energy tensor in the examples by Krasnikov must be ill-defined on the Cauchy horizon.

The organisation of this paper is as follows: In section II we present an independent derivation of the results of Krasnikov [7] by pointing out that the two-point function of a massless real scalar field in the conformal vacuum on two-dimensional Misner space can be given an image

sum form. In section III we use the fact that the conformal vacuum two-point function can be written as an image sum to prove that the stress-energy tensor in the conformal vacuum is ill-defined on the two-dimensional Misner space horizon. We then show that, as a consequence of the result which tells us that the stress-energy tensor in the conformal vacuum is ill-defined on the horizon, it must be similarly ill-defined for any (smooth) two-particle or thermal Hadamard state built on the conformal vacuum.

II. THE CONFORMAL VACUUM IMAGE SUM

Misner space (we mean here two-dimensional Misner space) is a well known solution of Einstein's vacuum field equations with topology $\mathbf{R} \times \mathbf{S^1}$, see e.g. [17] and references therein. It consists of an initial globally hyperbolic region IHG and a region of closed timelike curves CTC separated by a compactly generated Cauchy horizon (closed nullgeodesic) CH. A mathematical description of Misner space can be given by taking the metric on the initial globally hyperbolic region IGH to be

$$ds^2 = dt^2 - t^2 dx^2 \tag{1}$$

where $t \in (0, \infty)$ and x is periodic with period a. By going to double null coordinates (with an arbitrary length scale $L > 0$)

$$u := L \ln \frac{t}{L} - x \tag{2}$$

$$v := L \ln \frac{t}{L} + x \tag{3}$$

with $u, v \in \mathbf{R}$ and where points are identified by $(u,v) \leftrightarrow (u-a, v+a)$, one finds that the globally hyperbolic region IGH is conformal to the timelike cylinder, since

$$ds^2 = e^{\frac{u+v}{L}} du dv. \tag{4}$$

It is convenient to introduce another set of coordinates $U, V \in (-\infty, 0)$ defined by

$$U := -Le^{\frac{u}{L}} \tag{5}$$

$$V := -Le^{\frac{v}{L}}. \tag{6}$$

The metric then takes the form

$$ds^2 = dU dV \tag{7}$$

and points are identified by a Lorentz boost with rapidity a/L, that is $(U, V) \leftrightarrow (e^{-a/L}U, e^{a/L}V)$. The form of the metric and the identification of points under a Lorentz boost allows one to conclude that the initial globally hyperbolic region IGH of Misner space can be regarded as the quotient of the region $\{(U,V) : -\infty < U < 0, -\infty < V < 0\}$ of Minkowski space by a Lorentz boost. Furthermore, by extending the metric

over the Cauchy horizon CH and into the CTC region $\{(U,V) : -\infty < U < 0, 0 < V < \infty\}$ of Misner space one easily concludes that Misner space is the quotient of the region $\{(U,V) : -\infty < U < 0, V \in \mathbf{R}\}$ of Minkowski space by a Lorentz boost.

We now turn to consider the model of a real massless scalar field ϕ on the initial globally hyperbolic region IGH of Misner space satisfying the massless Klein-Gordon equation

$$\Box_g \phi = 0. \tag{8}$$

To quantise the system we use that the (conformal) positive frequency modes

$$u_k(u,v) = \begin{cases} (2a|k|)^{-1/2} e^{-ikv} & k < 0 \\ (2ak)^{-1/2} e^{iku} & k > 0 \end{cases} \tag{9}$$

with the (complexified) Klein-Gordon inner product defines a one-particle Hilbert space $\mathcal{H}$ which in turn defines a Fock space $\mathcal{F}$ with the conformal vacuum $|C\rangle$, see e.g. [18] for details on one-particle Hilbert space structure. From the standard mode-sum expansion of the field ϕ in terms of the positive frequency modes and by calculations with distributions we find that the conformal vacuum two-point function G_C in coordinates $x = (u, v)$ *formally* arises as the *image sum*

$$G_C(x, x') := \langle C | \phi(x) \phi(x') | C \rangle =$$

$$-\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \ln\{(u - u' + na)(v - v' - na)\}. \tag{10}$$

We remark that because of the problem with infrared divergences (from another point of view, because the argument of the logarithm in (10) should really be divided by an ill-defined length scale) in the two-dimensional massless scalar field, only derivatives of the two-point function G_C make sense. Hence it is to be understood that differentiation of the two-point function is performed before the summation over images. The stress-energy tensor $\langle T_{ab} \rangle_C$ in the conformal vacuum is defined by

$$\langle T_{ab} \rangle_C := \lim_{x, x' \to y} \mathcal{T}_{ab}(x, x') \tag{11}$$

where the (renormalised) *point-split stress-energy tensor* $\mathcal{T}_{ab}$ (here, thanks to flatness, primed and unprimed tensor indices can be identified) is defined by

$$\mathcal{T}_{ab}(x, x') := \{\partial_a \partial_{b'} - \frac{1}{2} g_{ab} g^{cd'} \partial_c \partial_{d'}\}$$
$$(G_C(x, x') - G_0(x, x')) \tag{12}$$

and where

$$G_0(x, x') = -\frac{1}{4\pi} \ln\{L^2(e^{\frac{u}{L}} - e^{\frac{u'}{L}})(e^{\frac{v}{L}} - e^{\frac{v'}{L}})\}$$

is the usual Minkowski vacuum two-point function. By straightforward calculations, one finds that the stress-energy tensor in the conformal vacuum in (u, v) coordinates is

$$\langle T_{ab}\rangle_C = -(\frac{\pi}{12a^2} + \frac{1}{48L^2\pi})\text{diag}(1,1) \qquad (13)$$

or by going to (U,V) coordinates

$$\langle T_{ab}\rangle_C = -(\frac{\pi L^2}{12a^2} + \frac{1}{48\pi})\text{diag}(U^{-2},V^{-2}). \qquad (14)$$

(in agreement with [7].)

The stress-energy tensor clearly diverges as one approaches the (future) Cauchy horizon ($V = 0, U = U_0 < 0$). Nevertheless, following Krasnikov [7] – but continuing to use a different method – we shall now show that there exist Hadamard states for which the stress-energy tensor vanishes on the initial globally hyperbolic region IGH. Consider a two-particle state in which the particles are in the momentum states $|n\rangle$ and $|-n\rangle$ then it is easy to show, using the mode sum expansion, that the two-point function G_2 in this state takes the form

$$G_2(x,x') = G_C(x,x') + 2Re\{u_n(x)u_n^*(x')\}$$
$$+2Re\{u_{-n}(x)u_{-n}^*(x')\}. \qquad (15)$$

Using (11) with G_2 replacing G_C in (12) we find that the stress-energy tensor $\langle T_{ab}\rangle_2$ in this two-particle state in (u,v) coordinates is given by

$$\langle T_{ab}\rangle_2 = -(\frac{\pi}{12a^2} + \frac{1}{48L^2\pi} - \frac{2\pi n}{a^2})\text{diag}(1,1) \qquad (16)$$

and thus we have reproduced the result of Krasnikov that (for any L) there exist choices of a and n such that the stress-energy tensor $\langle T_{ab}\rangle_2$ vanishes on the initial globally hyperbolic region IGH. Finally, consider a thermal state in which the thermal two-point function at inverse temperature β, G_β, is given by the image sum

$$G_\beta(x,x') = -\frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\{\ln(u - u' + na - i\beta m)$$
$$+ \ln(v - v' - na - i\beta m)\}. \qquad (17)$$

The thermal stress-energy $\langle T_{ab}\rangle_\beta$ is then obtained by using (11) with G_β replacing G_C in (12) which gives

$$\langle T_{ab}\rangle_\beta = -(\frac{\pi}{12a^2} + \frac{1}{48L^2\pi} - \frac{\pi}{2a^2})\sum_{m=1}^{\infty}\sinh^{-2}\frac{\pi m\beta}{a}\text{diag}(1,1) \qquad (18)$$

and thus we have reproduced the result of Krasnikov that (for any L) there exist choices of a and β such that the stress-energy tensor $\langle T_{ab}\rangle_\beta$ vanishes on the initial globally hyperbolic region IGH.

III. THE STRESS-ENERGY TENSOR IS ILL-DEFINED ON THE HORIZON

We shall now present an elementary proof of a result (the theorem below) which tells us that in the case of a massless real scalar field on two-dimensional Misner space the stress-energy tensor $\langle T_{ab}\rangle_C$ in the conformal vacuum is necessarily ill-defined on the Cauchy Horizon. Further, we shall show that, as a consequence of this result, the stress-energy tensor $\langle T_{ab}\rangle_2$ in the two-particle state and the thermal stress-energy tensor $\langle T_{ab}\rangle_\beta$ are also ill-defined on the Cauchy horizon. In particular, and this is the main point of the present paper, even for the states discussed in the previous section, for which the stress-energy tensor vanishes in the initial globally hyperbolic region IGH, it is still ill-defined *on* the Cauchy horizon (similarly to the results in [14,16]).

Theorem. Let b be any point on the Cauchy horizon CH, and N be any neighbourhood of b. Then there exists at least one-pair of non-null related points $(x,x') \in (N \cap IGH) \times (N \cap IGH)$, where IGH is the initial globally hyperbolic region, such that the (renormalised) point-split stress-energy tensor $\mathcal{T}_{ab}(x,x')$ in the conformal vacuum $|C\rangle$ is singular.

Proof. The (renormalised) point-split stress-energy tensor $\mathcal{T}_{ab}(x,x')$ is given by formulae (12)

$$\mathcal{T}_{ab}(x,x') =$$
$$\{\partial_a\partial_{b'} - \frac{1}{2}g_{ab}g^{cd'}\partial_c\partial_{d'}\}(G_C(x,x') - G_0(x,x'))$$

where the two-point functions G_C and G_0 in coordinates (U,V) are given by

$$G_C(x,x') = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\{\ln(L\ln(\frac{-U}{L}) - L\ln(\frac{-U'}{L}) + na)$$
$$+ \ln(L\ln(\frac{-V}{L}) - L\ln(\frac{-V'}{L}) - na)\}$$

and

$$G_0(x,x') = -\frac{1}{4\pi}\ln\{(U - U')(V - V')\}.$$

From the form of the metric (7) and the two-point functions G_C and G_0 it follows that the only non-vanishing components of the point-split stress-energy tensor are

$$\mathcal{T}_{UU}(x,x') = -$$
$$\frac{1}{4\pi}(\sum_n \frac{L^2}{UU'(L\ln(\frac{-U}{L}) - L\ln(\frac{-U'}{L}) + na)^2} - \frac{1}{(U - U')^2})$$

and

$$\mathcal{T}_{VV}(x,x') = -$$
$$\frac{1}{4\pi}(\sum_n \frac{L^2}{VV'(L\ln(\frac{-V}{L}) - L\ln(\frac{-V'}{L}) - na)^2} - \frac{1}{(V - V')^2}).$$

Clearly $N \cap IGH$ will contain an open rectangle $R = \{(U,V) : U_0 < U < U_1, V_0 < V < 0\}$. Thus it will be sufficient to show that there exists a pair of points in the open rectangle R having the properties stated in the theorem. Let $x = (U,V) \in R$. Then it is easy to see that

there exists a point $\hat{x} = (\hat{U}, \hat{V}) \in R$ spacelike separated from x such that every point $x_\delta = (\hat{U}, \delta)$, $\hat{V} < \delta < 0$ along the line connecting $\hat{x}$ to the point $(\hat{U}, 0)$ on the Cauchy horizon is also spacelike separated to x. We now show that we can choose δ so that the point-split stress-energy tensor $\mathcal{T}_{ab}(x, x')$ is singular. First we note that for any δ which takes the form $\delta = V e^{-ma/L}$ the mth term in the sum in the expression above for the component $\mathcal{T}_{VV}(x, x_\delta)$ will diverge and by taking m to be sufficiently large and positive we can arrange for δ to lie in the interval $(\hat{V}, 0)$. Taking $x' = x_\delta$ we have thus exhibited a pair of points in $(N \cap IGH) \times (N \cap IGH)$ such that the mth term in the sum of $\mathcal{T}_{VV'}(x, x')$ is singular. On the other hand, it is easy to see that the remaining terms of the sum are uniformly convergent in some neighbourhood of (x, x'). Thus we conclude that the point-split stress-energy tensor $\mathcal{T}_{ab}(x, x')$ is singular.

Corollary 1. The statement of the theorem is true for the two-particle point-split stress-energy tensor

$$\mathcal{T}_{ab}(x, x') =$$
$$\{\partial_a \partial_{b'} - \frac{1}{2} g_{ab} g^{cd'} \partial_c \partial_{d'}\}(G_2(x, x') - G_0(x, x')).$$

Proof. The proof is obvious from the facts that the two-point function $G_2(x, x')$ (15) is given by

$$G_2(x, x') = G_C(x, x')$$
$$+ 2Re\{u_n(x) u_n^*(x')\} + 2Re\{u_{-n}(x) u_{-n}^*(x')\}$$

and that the positive frequency modes $u_{\pm n}$ are smooth on IGH.

We remark that this corollary can in a trival way be extended to any many particle state built (by acting with creation operators smeared with smooth wave-functions) on the conformal vacuum, since the two-point function in such a state will consist of the conformal vacuum two-point function plus a smooth two-point function.

Corollary 2. The statement of the theorem is true for the thermal point-split stress-energy tensor

$$\mathcal{T}_{ab}(x, x') =$$
$$\{\partial_a \partial_{b'} - \frac{1}{2} g_{ab} g^{cd'} \partial_c \partial_{d'}\}(G_\beta(x, x') - G_0(x, x')).$$

Proof. The proof is immediate from the theorem and the observation that the thermal two-point function $G_\beta(x, x')$ (17) can be written as

$$G_\beta(x, x') = G_C(x, x') -$$
$$\frac{1}{4\pi} \sum_{m \neq 0} \sum_n \{\ln(L \ln(\frac{-U}{L}) - L \ln(\frac{-U}{L})' + na - i\beta m) +$$
$$\ln(L \ln(\frac{-V}{L}) - L \ln(\frac{-V'}{L}) - na - i\beta m)\}.$$

and since the differentiated form of the (double) sum is uniformly convergent (to a *smooth* limit).

It is clearly an immediate consequence of the above theorem and corollaries that the limit in equation (11), and in the corresponding equations for $\langle T_{ab} \rangle_2$ and $\langle T_{ab} \rangle_\beta$, is ill-defined, thus we have shown what we set out to show.

Finally we remark, that it is clear that a similar theorem holds in the context of massless automorphic fields on two-dimensional Misner space, since it can be shown [19] that the two-point function of a massless automorphic field takes the same form as the conformal two-point function G_C with the exception that an *automorphic* factor of $\exp(2\pi i \alpha n)$ is inserted into the image sum. (That the case of four-dimensional Misner space is similar was shown in [16].)

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Hadamard states and (in Section IIIB of [7]) an example is constructed of a true Hadamard state (of the massless scalar field on two dimensional Misner space) with bounded stress-energy tensor. We remark that it is easy to see that the statement of the theorem in this paper also holds for this latter example.

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